Chapter 7: Limit of a sequence

1. Sequences:

A sequence is a set of numbers $a_1, a_2, a_3, \ldots\ldots$ in a definite order of arrangement and formed according to a definite rule.

Definition1.1 A function of a positive integral variable, denoted by *f(n)* or a_n, where ***n*** = 1, 2, 3,…… is a called a sequence.

Each number in the sequence is called a ***term***; a_n is called the n^{th} term. ***Three*** usual notations as follows:

(1) a_1, a_2, a_3;

(2) a_n (n = 1,2,3,….)

(3) $\{a_n\}$

Ex. 1.1: The sequence $\{a_n\}$ is defined inductively by

$$a_1 = 2\sqrt{5}, \ldots, \quad a_{n+1} = \frac{1}{2}(a_n + \frac{5}{a_n}) \qquad (n=1,2,3,\ldots\ldots)$$

Express $\dfrac{a_{n+1}-\sqrt{5}}{a_{n+1}+\sqrt{5}}$ in terms of a_n. Hence find a_n.

Ex. 1.2: Determine the n^{th} term of the sequence $a_1 = 1$, $a_n = \dfrac{4a_{n-1}-9}{a_{n-1}-2}$ $\quad (n = 2,3,4,\ldots\ldots)$

2. Limit of a Sequence

We always interest about the sequence as for 'large' values of *n*. What are the 'very distant' members of the sequence are like.

For instance, $\{1-\frac{1}{n}\} = 0, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \ldots\ldots\ldots.$

For the above example, as *n* goes large, the term will tend to 1.
i.e. 1 is called the '***limit of the sequence***'.

The sequences $\{1-\frac{1}{n}\}$ = 0, 2, 0, 2,….has no such property. $\therefore$ it has no limit.

***Definition* 2.1** Let $\{a_n\}_{n=1}^{\infty}$ be a sequence and *L* is a fixed real number.

Then $\{a_n\}_{n=1}^{\infty}$ has the limit *L* as *n* tends to infinity iff

$[\forall e > 0, \exists N \in N \quad s.t. \quad |a_n - L| < e \quad \forall_n > N]$

We write $a_n \to L$ as $n \to \infty$ OR $\lim_{n\to\infty} a_n = L$

N.B.: (1) $|a_n - L| < e \Rightarrow -e < |a_n - L| < e$

$\Rightarrow L - e < a_n < L + e$

$\Rightarrow$ where the open interval (L - e, L+ e) is called the e - neighborhood of the limit *L*.

ε - is arbitrary chosen, that is the ε -neighborhood of *L* is not unique.

(2) *N* depends on ε

(3) x_n clusters around the ε - neighborhood of a limit '*a*' of a sequence

(4) $|a_n - L| < e$, $\forall n > N$ means $|a_n - L| < e$

$|a_{n+1} - L| < e$, $|a_{n+2} - L| < e$ |,…are also true.

i.e. $n > N$ is used to make sure that the tail of the sequence and it is of the sequence and it is often to say "for sufficiently large of *n*".

Ex. 2.1 Using the above definition to prove that

(a) $\lim_{n\to\infty} \frac{n}{n+1} = 1$

(b) $\lim_{n\to\infty} \left\{2 + (-\frac{1}{2})^n\right\} = 2$

Classworks: (a) Prove that if $0 < \alpha < 1$, then $\lim_{n\to\infty} \alpha^n = 0$

(b) Prove that if $a > 1$, then $\lim_{n\to\infty} \sqrt[n]{a} = 1$

(c) Prove that $\lim_{n\to\infty} \frac{C}{n^p} = 0$ where $C \neq 0$, & $p > 0$ are constants.

(d) Prove that $\lim_{n\to\infty} \frac{\sin \frac{np}{2}}{n} = 0$

Definition 2.2 A sequence $\{a_n\}_{n-1}^{\infty}$ is called a ***constant sequence*** iff there is a positive integral number s.t. $a_n = k$ for $n \geq np$.

For example: C_1 = {2, 2, 2,……}

C_2 = {-20, -10, 5, 4, 1, 1, 1, 1, …} are constant sequences.

***Definition* 2.3** If a sequence { a_n } has a limit zero, then this sequence is called an infinitesimally small quantity

i.e. $\lim_{n\to\infty} a_n = 0$ iff $\forall e > 0$, $\ni N \in \mathrm{N}$, s.t. $n > N$, s.t. $|a_n| < e$

***Theorem* 2.1** $\lim_{n\to\infty} a_n = L$ iff $\lim_{n\to\infty} (a_n - L) = 0$

***Theorem* 2.2** Let *q* is a fixed real number & $|q|$ <1, then $\lim_{n\to\infty} q^n = 0$

***Theorem* 2.3** Let *x* be a fixed positive numbers, then $\lim_{n\to\infty} \sqrt[n]{x} = 1$.

3. *Infinity*

***Definition* 3.1** A sequence $\{x_n\}_{n=1}^{\infty}$ tends to infinity as n tends to ∞ iff $\forall M>0$, $\exists N \in N$ s.t. when $n > N$, $|x_n| > M$.

We can write $\lim_{n\to\infty} x_n = \infty$ OR $x_n \to \infty$ as $n \to \infty$.

N.B.: (1) The tail of the sequence x_n lies outside the open interval $(-M, M)$.

(2) $\lim_{n\to\infty} x_n = \infty$ means the sequence diverges to infinity (i.e. the sequence is said to be divergent).

***Definition* 3.2** (a) A sequence $\{x_n\}_{n=1}^{\infty}$ tends to +ve infinity as $n\to\infty$, iff $\forall M>0$, $\exists N \in N$ s.t. $n > N$, $x_n > M$, denoted by $\lim_{n\to\infty} x_n = +\infty$ OR $x_n \to +\infty$ as $n\to\infty$.

(b) A sequence $\{x_n\}_{n=1}^{\infty}$ tends to –ve infinity as $n\to\infty$, iff $\forall M > 0$, $\exists \mathrm{N} \in \mathrm{N}$ s.t. $n > N$ $x_n < -M$, denoted by $\lim_{n\to\infty} x_n = -\infty$ OR $x_n \to -\infty$ as $n\to\infty$.

N.B.: If $\lim_{n\to\infty} x_n = \pm\infty$, then $\lim_{n\to\infty} x_n = \infty$. But conversely is not true i.e. If $\lim_{n\to\infty} x_n = \infty$, then x_n may not tend to +ve ∞ OR –ve ∞.

e.g. $\{1,-2,3,-4,\ldots\ldots\ (-1)^{n+1}n, \ldots..\}$

Ex. 3.1: By definition, show that $\lim_{n\to\infty} a_n = \infty$ where $a_n = 3^{2n-1}$.

***Theorem* 3.1** Let q be a fixed real number with $|q| > 1$, then $\lim_{n\to\infty} q^n = \infty$. Furthermore, if $q > 1$, then $\lim_{n\to\infty} q^n = +\infty$.

PROOF: Let M be any given +ve number

(i) If $M > 1$, then the inequality $|q^n| = |q|^n > M$

$$\Rightarrow n \ln|q| > \ln M$$

$$\Rightarrow n > \frac{\ln M}{\ln|q|}$$

$\therefore$ We take $N = \dfrac{\ln M}{\ln|q|}$ Then when n>N, we have $|q^n| > M$.

For details, read P. 283

***Theorem* 3.2** If $\lim_{n\to\infty} x_n = \infty$ & k is a non-zero constant (k is independent of n) then $\lim_{n\to\infty} kx_n = \infty$.

N.B. : (i) If $\lim_{n\to\infty} x_n = \lim_{n\to\infty} y_n = +\infty$, then

(a) $\lim_{n\to\infty} (x_n + y_n) = +\infty$.

(b) $\lim_{n\to\infty} kx_n = \begin{cases} +\infty & if\ k + ve \quad constant \\ -\infty & if\ k - ve \quad constant \end{cases}$.

(c) $\lim_{n\to\infty} (x_n y_n) = \infty$

(ii) If $\lim_{n\to\infty} x_n = \infty$ and $x_n \neq 0$, $\forall n \in \mathrm{N}$, then $\lim_{n\to\infty} \frac{1}{x^n} = 0$.

(iii) If $\lim_{n\to\infty} x_n = 0$ and $x_n \neq 0$, $\forall n \in \mathrm{N}$, then $\lim_{n\to\infty} \frac{1}{x^n} = \infty$.

Read P.284 for more details!!

4. Bounded and Unbounded Sequences

Definition* 4.1** A sequence $\{x_n\}_{n=1}^{\infty}$ is called ***bounded sequence iff $\exists M \in \Re$ s.t. $|x_n| \le M \quad \forall n = 1,2,3,\ldots\ldots$

E.g. $s_n = \{\sin \frac{np}{2}\}$

Definition* 4.2** $\{x_n\}_{n=1}^{\infty}$ is ***bounded above (***below***) iff $\exists M \in R$ s.t. $x_n \le (\ge) M \quad \forall n \in N$ and M is called the ***upper bound*** (***lower bound***).

N.B.: the upper bound and lower bound of a sequence are ***not unique***.

Theorem* 4.1** A ***convergent sequence is a ***bounded sequence***.

PROOF: Suppose $\{x_n\}$ converges to L (i.e. $\lim_{n\to\infty} x_n = L$)

$\therefore$ For any given $\varepsilon > 0$ (take $\varepsilon = 1$)

$\therefore \quad \forall e = 1 > 0$, $\exists N \in \mathrm{N}$ s.t. $|x_n - L| < e = 1$. $\forall n > N$

$\therefore \quad |x_n - L| < 1$

i.e. $-1 < x_n - L < 1$

$\therefore \quad L - 1 < x_n < L + 1 \qquad \forall n > N$

Take M = max ($|x_1|, |x_2|, \ldots\ldots |x_n|, |L-1|, |L+1|$), then $|x_n| \le M$. $\forall n = 1,2,3,\ldots\ldots$

Hence $\{x_n\}$ is bounded.

N.B.: The converses of the above theorem need not be true.
For instances {1,-1,1,-1,1,-1,…..} it is bounded but it has no limit.

***Theorem* 4.2** $\{x_n\}_{n=1}^{\infty}$ is called unbounded sequence iff $\exists M \in \Re^+$, $\exists n_o \in \mathrm{N}$ s.t. $|x_{n_o}| > M$.

N.B.: If $\lim_{n\to\infty} x_n = \infty$ (OR $\pm\infty$), then the sequence $\{x_n\}$ is unbounded. Conversely, an unbounded sequence may not tend to infinity. For instance, the sequence {1,0,2,0,3,0…} is unbounded, but it does not tend to infinity.

5. Properties of Limits of Sequences

***Theorem* 5.1** y_n then $\exists N \in \mathrm{N}$ s.t. $x_n > y_n \quad \forall n > N$ (Important condition: only the tail!!)

***Theorem* 5.2** **Uniqueness of Limit**
The limit of a convergent sequence is unique.

***Theorem* 5.3** **Principle of Sequeezing OR Sandwich Theorem**
If $x_n \le y_n \le z_n$, $\forall_n > N$ Then if $\lim_{n\to\infty} x_n = \lim_{n\to\infty} z_n = \ell$

Then $\lim_{n\to\infty} y_n = \ell$.

PROOF:

Let ε be any given +ve number. For this ε, there is a positive integer N_1, s.t.

$a - e < x_n < a + e$, $\forall_n > N_1$………(1)

Again, for this ε, there is another positive integer N_2, s.t.

$a - e < z_n < a + e$, $\forall_n > N_2$………(2)

Take N_3 = max (N, N_1, N_2) then n > N_3.

Inequality (1), (2) & $x_n \le y_n \le z_n$ all hold simultaneously.

Hence $a - e < x_n \le y_n \le z_n < a + e$, $\forall n > N_3$.

$\therefore\ |y_n - a| < e\ ,\ \ \forall n > N_3\ .$

$\therefore\ \lim_{n\to\infty} y_n = a$

N.B.: The Squeezing principle is a very useful tool for evaluation of the limit of a given sequence.

Ex. 5.1: Find the limit of (a) $\lim_{n\to\infty}\left(\frac{1}{n^2} + \frac{1}{(n+1)^2} + \dots\dots\dots + \frac{1}{(2n)^2}\right)$

(b) Prove that $\lim_{n\to\infty}\frac{\sin n}{n} = 0$

Ex.5.2: Let a be a real number >1. Prove that $\lim_{n\to\infty}\frac{n}{a^n} = 0$

Ex.5.3: Find the $\lim_{n\to\infty}\sqrt[n]{n}$

6. Operations of Limit of Sequences

***Theorem* 6.1**

If $\lim_{n\to\infty} x_n = \ell_1$, $\lim_{n\to\infty} y_n = \ell_2$ then

(1) $\lim_{n\to\infty}(x_n \pm y_n) = \ell_1 \pm \ell_2$

(2) $\lim_{n\to\infty}(x_n y_n) = \ell_1 \cdot \ell_2$

(3) $\lim_{n\to\infty}\left(\frac{x_n}{y_n}\right) = \frac{\ell_1}{\ell_2}$ for $\ell_2 \neq 0$ & $y_n \neq 0$

(4) $\lim_{n\to\infty} kx_n = k\lim_{n\to\infty} x_n = k\ell_1$ where k is a constant

(5) If $|x_n|$ is convergent, then $\lim_{n\to\infty}|x_n| = |\ell_1|$

N.B.: The converse is not true!!

$\lim_{n\to\infty}|x_n| = |\ell_1|$ not $\Rightarrow$ $\lim_{n\to\infty} x_n = \ell_1$

e.g. $\lim_{n\to\infty}|(-1)^n| = 1$ but $\lim_{n\to\infty}|(-1)^n|$ does not exist.

To apply (1) to (5), you must first check the limit of each term exists.

Ex. 6.1: Find $\lim_{n\to\infty}\frac{n^2 - n + 2}{3n^2 + 2n + 4}$

***Theorem* 6.2**

Let $\{x_n^{(1)}\}_{n=1}^{\infty}$, $\{x_n^{(2)}\}_{n=1}^{\infty} \dots \{x_n^{(k)}\}_{n=1}^{\infty}$ be m convergent sequences.

Where m is a fixed +ve integer & let $\lim_{n\to\infty} x_n^{(k)} = \ell_k$

$\therefore$ (i) $\lim_{n\to\infty}\{\sum_{k=1}^{m} x_n^{(k)}\} = \sum_{k=1}^{m}[\lim_{n\to\infty} x_n^{(k)}] = \sum_{k=1}^{m}\ell_k$

(ii) $\lim_{n\to\infty}\{\prod_{k=1}^{m} x_n^{(k)}\} = \prod_{k=1}^{m}[\lim_{n\to\infty} x_n^{(k)}] = \prod_{k=1}^{m}\ell_k$

***Theorem* 6.3**

If $\{x_n\}$ is bounded & $\lim_{n\to\infty} y_n = 0$ then $\lim_{n\to\infty}(x_n y_n) = 0$

e.g. Find $\lim_{n\to\infty}\frac{(-1)^n}{n}$

***Theorem* 6.4**

If $\{x_n\}$ is bounded & $\lim_{n\to\infty} y_n = \infty$, then $\lim_{n\to\infty}(x_n + y_n) = \infty$

e.g. Find $\lim_{n\to\infty}(n + \sin n)$

e.g. Find the limit of (a) $\lim_{n\to\infty}(\sin n + \frac{n^3}{\sqrt{n^2+1}})$

(b) $\lim_{n\to\infty}[n+(-1)^n(\frac{1}{1.3}+\frac{1}{3.5}+\text{.........}\ \ldots+\frac{1}{(2n-1)(2n+1)}]$

7. Some worked examples:

Ex. 7.1: Show that the limit value $\lim_{n\to\infty}(\frac{1}{n}-\frac{2}{n}+\frac{3}{n}-\frac{4}{n}+\text{.......}+(-1)^{n-1}\frac{n}{n})$ does not exist.

Ex. 7.2: (1) Prove that $\lim_{n\to\infty}\frac{a^n}{n!}=0 \quad \forall a\in\Re$

(2) Prove that for any +ve real numbers a & b, $\lim_{n\to\infty}[(a_n+b)^{\frac{1}{n}}-1]=0$

(3) Find the $\lim_{n\to\infty}(\frac{1}{n^2}+\frac{1}{(n+2)^2}+\ldots+\frac{1}{(2n)^2})$

(4) Find $\lim_{n\to\infty}(\frac{1}{\sqrt{n+1}}+\frac{1}{\sqrt{n+2}}+\text{.........}+\frac{1}{\sqrt{n+n}})$

Ex.7.3: Let $A\in\Re^+$ and $\{a^n\}$ be a sequence of real numbers s.t. $a_1\ge A$ & $a_{n+1}=\frac{1}{2}(a_n+\frac{A^2}{a_n})$ for $n>1$.

(a) Show that $a_n\ge A \quad \forall n\in\Re^+$, hence show that $a_n\ge A\le\frac{1}{2}(a_{n-1}-A)$

(b) Find $\lim_{n\to\infty}a^n$.

Ex.7.4: A sequence $\{a_n\}$ is defined by $a_1=4$ and $a_{n+1}=\frac{6a_n^2+6}{a_n^2+11}$ for $n\ge1$.

(a) Prove, by induction, that $a_n>3$, $\forall n\in\Re$

(b) Show that for any +ve integers n, $\frac{a_{n+1}-3}{a_n-3}<\frac{9}{10}$

(c) Find $\lim_{n\to\infty}a_n$.

Ex.7.5: A sequence $\{x_n\}$ of real numbers is defined by $x_{n+1}=\frac{x_n+x_{n-1}}{2}$ for $n>1$.

(a) Show that for $n>1$, $x_n-x_{n-1}=(-\frac{1}{2})^{n-1}(x_1-x_0)$.

(b) Find $\lim_{n\to\infty}a_n$.

Ex.7.6: The sequence $\{x_n\}$ is defined by $x_{n+1}=x_n^2-2x_n+2$ for $n\ge1$.

(a) Show that $x_{n+1}-1=(x_1-1)^{2^n}$ for $n\ge1$.

(b) Find the value of $\lim_{n\to\infty}x_n$ for the following cases:

(i) $1<x_1<2$

(ii) $x_1=2$

(iii) $x_1>2$.

Ex.7.7: Find two diverging sequences $\{x_n\}$ & $\{y_n\}$ s.t.

(a) $\{x_n\pm y_n\}$ is convergent;

(b) $\{x_ny_n\}$ is convergent.

8. Monotonic Sequences

***Definition* 8.1** $\{x_n\}_{n=1}^{\infty}$ is said to be

(i) Monotonic increasing iff $x_{n+1} \geq x_n$ $\forall n = 1,2,3\ldots.$

(strictly increasing iff $x_{n+1} > x_n$)

(ii) Monotonic decreasing iff $x_{n+1} \leq x_n$ $\forall n = 1,2,3\ldots.$

(strictly decreasing iff $x_{n+1} < x_n$)

(iii) Monotonic iff (i) OR (ii). $\forall n = 1,2,3,\ldots$

***Theorem* 8.1** (a) $\{x_n\}_{n=1}^{\infty}$ is monotonic increasing & bounded above (unbounded $\Rightarrow \lim_{n\to\infty} x_n = +\infty$) $\Rightarrow$ has limit.

(b) $\{x_n\}_{n=1}^{\infty}$ is monotonic decreasing & bounded below (unbounded $\Rightarrow \lim_{n\to\infty} x_n = -\infty$) $\Rightarrow$ has limit.

Ex. 8.1:

Let a be a positive real number. A sequence $\{y_n\}$ is defined by $y_1 = \sqrt{2}$ and $y_n = \sqrt{2 + y_{n-1}}$, for $n > 1$

(a) Show that $\{y_n\}$ is monotonic increasing;

(b) Show that $\{y_n\}$ is bounded from above;

(c) Find $\lim_{n\to\infty} y_n$.

Ex. 8.2:

Let a and b be 2 real numbers s.t. $a > b > 0$. Two sequences $\{a_n\}$ and $\{b_n\}$ are defined by $a_n = \dfrac{a_{n-1} + b_{n-1}}{2}$,

$b_n = \sqrt{a_{n-1}b_{n-1}}$ for $n > 1$ & $a_1 = \dfrac{a+b}{2}$; $b_1 = \sqrt{ab}$.

(a) Prove that $\{a_n\}$ is monotonic decreasing. Hence deduce that $\{b_n\}$ is monotonic increasing.

(b) Prove that $\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n$.

Classwork Ex.:

(1) Let $\{a_n\}$ be a sequence is defined by $a_1 = 1$ & $a_n = \dfrac{a_{n-1}}{a_{n-1+1}}$ for $n > 1$, show that $\{a_n\}$ is convergent & find its limit.

(2) A sequence $\{x_n\}$ is defined by $x_n = 1 + \dfrac{1}{2^2} + \dfrac{1}{3^2} + \ldots\ldots + \dfrac{1}{n^2}$ for $n \geq 1$

(a) By using the result $\dfrac{1}{n^2} \leq \dfrac{1}{n(n-1)}$ $\forall n > 1$,

Show that $\{x_n\}$ is bounded from above.

(b) Show that $\{x_n\}$ is convergent.

(3) $\lim_{n\to\infty} (\dfrac{2}{n} + \dfrac{4n}{4n-1})$

(4) $\lim_{n\to\infty} (\sqrt{n+1} - \sqrt{n})$

(5) $\lim_{n\to\infty} (n - \dfrac{n^2}{n+1})$

(6) $\lim_{n\to\infty} (\dfrac{1}{n^2} + \dfrac{2}{n^2} + \ldots + \dfrac{n}{n^2})$

(7) $\lim_{n\to\infty} \dfrac{(3n+1)(2n^2-3)(2-n)}{n^3+1}$

(8) $\lim_{n\to\infty} n[\sqrt{1+\frac{1}{n}}-\sqrt{1-\frac{1}{n}}]$

(9) $\lim_{n\to\infty} (\sqrt[3]{n+1}-\sqrt[3]{n})$

(10) $\lim_{n\to\infty} \frac{2^n+3^n}{2^n-3^n}$

(11) $\lim_{n\to\infty} [\frac{n}{n^2-2}+\frac{4^n(-1)^n}{2^n-1}]$

(12) $\lim_{n\to\infty} \frac{1+\frac{1}{2}+\frac{1}{2^2}+...+\frac{1}{2^n}}{1+\frac{1}{3}+\frac{1}{3^2}+...+\frac{1}{3^n}}$

(13) $\lim_{n\to\infty} [\frac{1}{1.2}+\frac{1}{2.3}+.......+\frac{1}{n(n+1)}]$

(14) $\lim_{n\to\infty} (\sum_{k=1}^{n}\frac{1}{1+2+...+k})$

9. An important limit: The number e

Prove: The sequence $a_n=(1+\frac{1}{n})^n$ is monotonic increasing & bounded above by 3 & hence it is convergent.

***Definition* 9.1** The limit of the sequence $\{a_n\}$ $\lim_{n\to\infty}(1+\frac{1}{n})^n=e$ (exponential number).

i.e. As n→∞, then $(1+\frac{1}{n})^n=1+\sum_{r=1}^{n}\{\frac{1}{r!}\prod_{k=0}^{r-1}(1-\frac{k}{n})\}\to 1+\sum_{r=1}^{\infty}(\frac{1}{r!})$, it suggests that

$$e=\lim_{n\to\infty}(1+\frac{1}{n})^n=1+\frac{1}{1!}+\frac{1}{2!}+.....$$
$$=2.71828....$$

Worked examples:

1. Show that (i) $\lim_{n\to\infty}(1+\frac{1}{n})^{xn}=e^x$

   (ii) $\lim_{n\to\infty}(1+\frac{x}{n})^n=e^x$

2. Evaluate the following limits:

   (i) $\lim_{n\to\infty}(1+\frac{1}{2^n})^n$

   (ii) $\lim_{n\to\infty}(\frac{n}{1+n})^n$

   (iii) $\lim_{n\to\infty}(\frac{n^2+1}{n^2})^{n^2+1}$

3. Prove the followings:

   (a)(i) $n!<(\frac{n+1}{2})^n$, $n>1$

   (ii) $(\frac{n}{e})^n<n!<e(\frac{n}{2})^n$

   (b) Using (ii) prove that $\lim_{n\to\infty}\frac{n!}{n^n}=0$